Application of Graph Matching

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Introduction

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Weighted Matching in $O(E\sqrt{V} \times |WORD|)$ time *assuming a reasonable arithmetic model* (Duan, Pettie, Su. 2017)

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Approximate (weighted) matching for dynamic graph is an active research topic, with different complexity and approximation ratio considered.

Yes, this is very deep. Let's not get in further :)

Chinese Postperson Problem

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Given a weighted graph, find a minimum-weight circuit that visits **every edges** at least once.

This resembles the *Traveling salesperson problem*, where we should visit **every vertices**.

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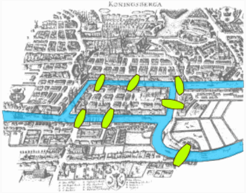

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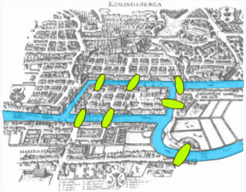

Theorem (Euler circuit). In a connected graph, there exists a circuit which visits all edges exactly once if and only if every vertex have even degree.

Proof by induction on $|E(G)|$ is somewhat well-known.

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Change the goal! Rather than finding a circuit, *duplicate* the edges to make the graph to have a Euler circuit.

We should **minimize** the total weight of the augmented graph.

Chinese Postperson Problem

Given a graph, duplicate the edges to make the graph Eulerian, while minimizing the total edge weight.

After the duplication, every edge should have even degree.

There are even number of odd degree vertices. Suppose there are two of them. Can we solve it?

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We can duplicate all edges in shortest path.

We can't do better: Set of duplicated edges should connect two odd-degree vertices.

Case of two odds: Find the shortest path between two, and duplicate along the path.

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In general, this works in matching framework. For each pair of odd-degree vertices, we can assign the weight as a length of shortest path, and find the optimal matching.

Using Floyd-Warshall and algorithm from Gabow gives $O(N^3)$ solution.

This certainly gives the solution, but is this optimal?

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Take any nontrivial component of it. It should have two *leaf* vertices (degree 1).

Take the unique path, remove it, and repeat it.

Now we decomposed this set into the disjoint path between some pair of odd-degree vertices. Our solution can't be worser than that.

Christofide’s Algorithm

Christofide's Algorithm

Although the Chinese postperson problem was solved in polynomial-time, it's vertex variant, the *Traveling salesperson problem* can't be solved.

But we can try to **approximate**: Can we find a solution that is close to the optimal solution?

*W*hat is *close enough* solution? We use the notion that the algorithm returns a solution that is always not worser than the optimal solution by some multiplicative factor.

For example, we will talk about an algorithm that returns a solution that is **never worser** than 1.5 times the optimal solution.

But it only works in a special case: The distance should form a *metric space*.

Christofide's Algorithm

In *traveling salesperson problem*, we should visit every vertices and minimize the distance of total tour.

Let's try to make a following assumption on the *distance between two points*.

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) + d(y, z) \geq d(x, z)$

Corollary. If those three axioms holds, then $d(x, y) \geq 0$.

A distance function is a *metric*, if those three axioms hold. Those assumption hold on $\mathbb{R}^n$, shortest path in graph, etc.

Christofide's algorithm requires the distance function to be a metric.

Christofide's Algorithm

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We obtained a TSP tour which have at most $2 \times$ (MST weight) $\leq 2 \times OPT$ (OPT denotes size of optimal solution).

In other words, this is a **2-approximation** algorithm for metric TSP.

Christofide's Algorithm

So, we have a MST, we want an Euler tour, so we duplicated the edge..
But we've talked tirelessly about this stuff!

Rather than naively duplicating, let's gather all odd-degree vertices in MST and find a minimum weight matching.

Add the matching to the tree. The graph is Eulerian. Do the same thing. Much better, but how much?

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Lemma. The added min-weight matching has weight at most $0.5 \times OPT$

Proof. Take a TSP with weight OPT. Pull out all even-degreed vertex (By axiom 3 this decreases the length). Now you have a tour that visits all odd-degree vertices and only that.

Now, take the even-indexed edges, and take the odd-indexed edges. Both are a perfect matching: If the lemma don't hold, their total length will be greater than OPT.

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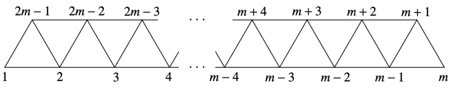
One of the worst case (where 1.5 is achieved) looks like the following:



Round trip through the faces uses $2m-1$ edges.

However, if your MST is unluckily a path from $1, 2, \dots, m$, along with all edges $i, 2m-i$, then the matching will spend extra $m-1$ edges.

If you think that's too unlucky, perturb slightly to make it a unique MST.

Parity shortest paths

You are given a positive-weighted undirected graph.

A problem of finding a shortest walk between two vertices s, t are already considered. Shortest walk are necessarily simple (So, they are shortest path).

Finding a shortest *odd* or *even* walk can be done similar. By odd/even, it concerns about the parity of number of edges.

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Finding a shortest *odd* or *even* walk can be done similar. By odd/even, it concerns about the parity of number of edges.

Create a copy of graph G, G', and for all edge $e = \{u, v\}$, connect $G(u) \leftrightarrow G'(v)$, $G(v) \leftrightarrow G'(u)$.

Then, any path between G, G' have odd length, and between themselves have even length. So you can find a path between $G(s), G(t)$ or $G(s), G'(t)$.

Are they simple walks in reality?

Parity shortest paths

Why should I care? Actually I'm not sure, but we can agree that they are pretty fundamental questions to ask in graphs.

For example, if you can find a shortest even/odd path between s, t, then you can find a shortest even/odd cycle in an undirected graph, by enumerating all $e = \{u, v\}$ and finding a shortest parity path between u, v in $G - e$.

You know, odd cycles define bipartite graphs.

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What about directed graph? You can't even find the **existance** of parity path in polynomial time (Thomassen 1985).

But, shortest odd cycle in directed graph are easy to find.

Existance of even cycle can be found in polynomial time, using some fancy algebra, done by fancy researchers not long ago (Robertson, Seymour, Thomas 1999)

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Also, connect $G(v) \leftrightarrow G'(v)$ for all $v \in V(G)$.

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Then, if a perfect matching exists, s and t is connected by an even-length path. We can see the existance.

Why is it true? Note that each edges are selected at most once. Otherwise, we discard both and connect $G(v), G'(v)$ instead.

Among the selected edges, there exists two odd-degree vertices. They should belong in same connected component, which is a path (degree is at most 2!)

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Generalize with weights: Give the edges their weight, and connect $G(v) \leftrightarrow G'(v)$ for all $v \in V(G)$ with cost 0.

Minimum weight perfect matching gives exactly the shortest path. Cycles doesn't exist because they are not cost-efficient.