Matroid Intersection

Jaehyun Koo (koosaga.com)

Introduction

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Recall: For matroid $M = (E, I)$, Greedy algorithm can compute the largest independent set (base), along with the weighted version, with $|E|$ **oracle calls**.

The **oracle** is an *algorithm* that determines whether $X \in I$.

Consider two matroids from the same ground set
$M_1 = (E, I_1), M_2 = (E, I_2)$.

Can we devise an **efficient algorithm** that computes a largest **common** independent set for both matroid?

What about the one with largest weight?

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Fortunately, this is not true for today's topic.

We discussed a bit about the *Maximum Matching* problem in Lecture 2.
We will show how this is related to matroid intersection.

Maximum Matching

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Given a bipartite graph $G = (L, R, M)$, find a largest set of edges that does not share an endpoint.

In other words, you should select a maximum size subset of edges such that

1. For each node $v \in L$, you should select at most one edge incident with v.
2. For each node $w \in R$, you should select at most one edge incident with w.

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Set of matchings in bipartite graph is a intersection of independent set in two partition matroids.

We can compute the largest common independent set with **polynomial** number of oracle calls.

Arborescence

Connected graphs have spanning trees.

Minimum spanning tree can be solved in polynomial time by applying matroid greedy algorithm. (Basically, this is *Kruskal's algorithm*.)

Let's consider a directed graph counterpart.

Given a directed graph and a **source vertex** v, *find a directed tree with minimum cost such that every edge is directed out of* v.

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This is sufficient.

You can find the minimum weight common base of graphic matroid and partition matroid.

Hamilton Path

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... Not really, then cycles also are.

Paths are an acyclic connected(single-component) subset of edge, where each vertex in path have at most one outgoing and ingoing edges. This is sufficient.

Hamilton Path

Outgoing and ingoing edges form a partition matroid.

Acyclic edges form a graphic matroid.

Connectivity automatically follows if the path have length $N - 1$.

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Theorem (Koo, 2020). Hamiltonian path is in P, thus $P = NP$.

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You **CANNOT** find the intersection of three matroid in polynomial time.

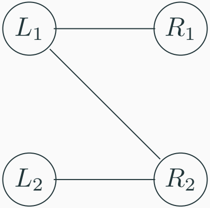
But why? You can intersect M_1, M_2 *and then intersect it again with* M_3

The intersection of two matroid is not a matroid.

Bipartite matching is intersection of two matroid, but not a matroid.

Hamilton Path

Bad things happen if you first add (L_1, R_2) in your greedy algorithm.



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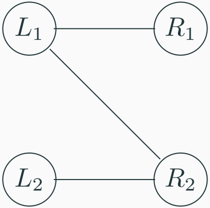

I can't believe it! Where is my million dollars? Surely the algorithm could be extended!

Intersection

Matroid Intersection Theorem

Theorem. Let $M_1 = (E, I_1), M_2 = (E, I_2)$ be matroids with rank functions r_1, r_2 respectively. Then,
$\max_{I \in I_1 \cap I_2} |I| = \min_{X \subseteq E}(r_1(X) + r_2(E - X))$

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$\max_{I \in I_1 \cap I_2} |I| = \min_{X \subseteq E}(r_1(X) + r_2(E - X))$

Yay, min-max theorems!

Note that $\leq$ is easy: $|I \cap X| = r_1(I \cap X) \leq r_1(X)$, and likewise.

$\geq$ is very hard and complicated.

Matroid Intersection Theorem

Well but actually no, there is a proof that fits in one page.

Proof. (Woodall) Let k be the minimum obtained. Let $x \in E$ be such that $r_1(\{x\}) = r_2(\{x\}) = 1$. (If there is no such x, then it is trivial.)

Let $Y = E \setminus \{x\}$. We may assume that $M_1 \setminus x$ and $M_1 \setminus x$ have no common independent set of size k. Thus, by induction,

$$r_1(A_1) + r_2(A_2) \leq k - 1$$

for some partition (A_1, A_2) of Y. Moreover M_1/x and M_2/x have no common independent set of size $k-1$. Thus,

$$r_1(B_1 \cup \{x\}) - 1 + r_2(B_2 \cup \{x\}) - 1 \leq k - 2$$

for some partition (B_1, B_2) of Y. By the submodularity,

$$r_1(A_1 \cap B_1) + r_1(A_1 \cup B_1 \cup \{x\}) \leq r_1(A_1) + r_1(B_1 \cup \{x\}),$$
$$r_2(A_2 \cap B_2) + r_2(A_2 \cup B_2 \cup \{x\}) \leq r_2(A_2) + r_2(B_2 \cup \{x\}).$$

However, $r_1(A_1 \cap B_1) + r_2(A_2 \cup B_2 \cup \{x\}) \geq k$ and $r_1(A_2 \cap B_2) + r_2(A_1 \cap B_1 \cup \{x\}) \geq k$. A contradiction. □

But this is outside of our focus: We will talk about the **constructive** proof that also gives an **algorithm** for a matroid intersection.

Matroid Intersection Theorem

Given $I \in I_1 \cap I_2$, we find $J \in I_1 \cap I_2$ such that $|J| = |I| + 1$ or find a certificate X which shows it is impossible.

The certificate is the set X such that $|I| = r_1(X) + r_2(E - X)$.

Consider a directed graph G with vertex set E, and the edge set as a union of

1. $A_1(I) = \{(z, y) | y \in E \setminus I, z \in I, I + \{y\} - \{z\} \in I(M_1)\}$
2. $A_2(I) = \{(y, z) | y \in E \setminus I, z \in I, I + \{y\} - \{z\} \in I(M_2)\}$

(Basically, they model the "exchange" or "trade" step: When we have a set that is independent for only one, we remove one and try to fit in the other set: hopefully both.)

Matroid Intersection Theorem

Let $X_1, X_2 \subseteq V(G)$ be a set such that

1. $X_1 = \{x \in E \setminus I | I + \{x\} \in I_1)\}$
2. $X_2 = \{x \in E \setminus I | I + \{x\} \in I_2\}$

When $X_1 \cap X_2$ is nonempty, we are *very happy*.

But life is hard. Suppose not. We should try to add X_1, which make $I \in I_1$ but not in I_2.

Then we take the edge A_2 to make $I \in I_2$ but not in I_1. And we take the edge A_1...

If we **find a path** from X_1 to reach X_2, then we are *happy*.

Matroid Intersection Theorem

Direction 1. If there is no path from $X_1 \to X_2$ in G, we can find the counterexample X.

If one of X_1, X_2 is empty, then I is already a base. Thus, assume both are nonempty.

Let X be a set that can reach X_2 in G. We can easily see that $X_1 \cap X = \emptyset, X_2 \subseteq X$, and there is no edge entering X. Then

1. $r_1(X) \leq |I \cap X|$. Suppose $r_1(X) > |I \cap X|$, then there exists some $y \in X - I$ such that $(I \cap X) + \{y\} \in I(M_1)$. Since $y \in X$, $y \notin X_1$ so $I + \{y\} \notin I(M_1)$. So there is some $z \in I - (I \cap X)$ such that $I + \{y\} - \{z\} \in I(M_1)$. This means there is edge (z, y) entering X, contradiction.

Matroid Intersection Theorem

1. $r_1(X) \leq |I \cap X|$. Suppose $r_1(X) > |I \cap X|$, then there exists some $y \in X - I$ such that $(I \cap X) + \{y\} \in I(M_1)$. Since $y \in X$, $y \notin X_1$ so $I + \{y\} \notin I(M_1)$. So there is some $z \in I - (I \cap X)$ such that $I + \{y\} - \{z\} \in I(M_1)$. This means there is edge (z, y) entering X, contradiction.
2. $r_2(E - X) \leq |I \cap (E - X)|$. Suppose $r_2(E - X) > |I \cap (E - X)|$, then there exists some $y \in (E - X) - I$ such that $(I \cap (E - X)) + \{y\} \in I(M_2)$. Since $y \in E - X$, $y \in E - X_2$, So $I + \{y\} \notin I(M_2)$. So there is $z \in I - (I \cap (E - X))$ such that $I + \{y\} - \{z\} \in I(M_2)$ (note that $z \in X$). This means there is edge (y, z) entering X, contradiction.

Honestly step 2 is just same as step 1. I just added it for completeness.

Matroid Intersection Theorem

Direction 2. If P is a **shortest** $X_1 - X_2$ path in G, then $I' = I \Delta V(P)$ is in $I_1 \cap I_2$.

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Let's skip the proof for a while, and see what is the implication.

$|I \Delta V(P)| = |I| + 1$, so we can start from the empty set, construct the graph, and find a shortest path to increase the size of common independent set, until we surely can't continue by direction 1.

We need polynomial number of oracle calls, and a simple shortest path search. **We have an algorithm!**

Wait, but we didn't proved Direction 2...

Lemma 1. Let $G = (X, Y, E)$ be a bipartite graph with **unique perfect matching** N. Then, we can label the vertices of X, Y as $x_1, x_2, \ldots, x_t$ and $y_1, y_2, \ldots, y_t$ respectively, such that $N = \{(x_1, y_1), (x_2, y_2) \ldots, (x_t, y_t)\}$, and $(x_i, y_j) \notin E$ for all $i < j$.

Proof. We use induction on t. If $t = 0$ it is obvious.

Note that there exists some vertex with degree one. Let's find any vertex v and find any "alternating walk": Odd-indexed edge are in N and Even-indexed are not. If the walk ends up finding a visited vertex, then the cycle induced by it breaks uniqueness. The walk will never end up with edges not in N because we can always extend (like geography game). So we find a odd-length path (not walk at this point) starting and ending with edges from N. The last vertex in the path have odd degree.

Now let (x, y) be an edge in N where x or y have degree one. Remove x, y and graph and find some ordering $\{(x_2, y_2), \ldots, (x_t, y_t)\}$. Say x is the vertex with degree 1. Then it's only connected with y, so $(x_1 y_j) \notin E$ for all $1 < j$. If y is such vertex proceed similarly.

Matroid Intersection Theorem

I won't explain all these stuffs. If you are interested, check it out. (You can google for better proofs too)

Lemma 2. Let $M = (E, I)$ be a matroid. Let I be an independent set in M, and let J be some subset S such that $|I| = |J|$. If there is a **unique bijection** $\alpha : I - J \to J - I$ such that $I - \{e\} + \{\alpha(e)\} \in I$. Then J is an independent set.

Proof. Let $G = (I - J, J - I)$, and it's edge set captures the exchange as above. By lemma 1 we can find some ordering $N = \{(y_1, z_1), \ldots, (y_t, z_t)\}$. Suppose J is has a circuit C and i be a smallest index such that $z_i \in C$. Consider any element $z_j \in C - z_i$. Since $i < j, (y_i, z_j) \notin D_M(I)$. So for all $z \in C - \{z_i\}$, $z \in span(I - y_i)$. (because, if $z \in I \cap J$ it's trivial, and otherwise $z_j \in span(I - y_i)$). So, $C \subseteq span(C - z_i) \subseteq span(I - y_i)$. Thus $z_i \in span(I - y_i)$. Contradiction.

Proof of Direction 2. Let $P = \{y_0, z_1, y_1, z_2, \ldots, z_t, y_t\}$ be a shortest path from X_1 to X_2. Let $J = \{y_1, \ldots, y_t\} \cup (I \setminus \{z_1, \ldots, z_t\})$ (which is the matroid we want to acquire, except y_0). Then $J \subseteq E, |J| = |I|$. and the arcs from $\{z_1, z_2, \ldots\} \to \{y_1, y_2, \ldots\}$ form a unique perfect matching. (If not: it's not a shortest path by following shortcut that exists) So $J \in I_1$. Also, $y_i \notin X_1$ for $i \geq 1$ because otherwise it's not shortest. So, $y_i + I \notin I_1$, and thus $r_1(I \cup J) = r_1(I) = r_1(J) = |I| = |J|$. (Their span is same) Since $I + y_0 \in I_1, J + y_0 \in I_1$. Similarly, $y_i \notin X_2$ for $i < t$. $y_i + I \notin I_2$. and similarly $I + y_t \in I_2, J + y_t \in I_2$. The proof is complete.

Weighted Matroids

And yes, you can also do the similar things in weighted variant.

Weight the vertices with their respective cost: For $e \in I$, the weight is $-w(e)$, otherwise, the weight is $w(e)$, and find the maximum weight path of $X_1 - X_2$.

If there are multiple such minimize the number of edges used.

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As far as I found, no introductory books and slides contain a proof for this algorithm. So I also won't.

Union

Matroid Union

Unlike intersection, union does not find $I \in I_1 \cup I_2$, which is trivial.

Instead, it's more like matroid **partitioning**.

Let $M_1, M_2, \cdots, M_n$ be a matroids in E. Let I_i be a set of independent sets of M_i. Let $I = \{J_1 \cup J_2 \cup \cdots J_n : J_i \in I_i\}$.

Theorem 1. $M = (E, I)$ is a matroid.

Theorem 2. Rank function of M is
$r_M(X) = min_{Y \subseteq X}(r_1(Y) + r_2(Y) + \cdots + r_n(Y) + |X - Y|)$

Matroid Union. Proof of Theorem 1

Due to time constraint we are not proving this. (But it's not super hard)

(I1, I2) trivial

(I3) Suppose $X, Y \in I, |X| < |Y|$. $X = \bigcup_{i=1}^{n} I_i, Y = \bigcup_{i=1}^{n} J_i$. We may assume I_i, J_i are independent, and I is formed with disjoint sets, so do J. There exists some index such that $|I_i| < |J_i|$. Since M_i is a matroid, there is $e \in J_i - I_i$ such that $I_i \cup \{e\}$ is independent in M_i. If $e \notin X$, then $X \cup \{e\} \in I$ so (I3) holds. So assume $e \in I_j$ for some $j \neq i$. We define $I_k = (I_k - \{e\}$ if $k = i$, $I_k + \{e\}$ if $k = j$. I_k) otherwise. Then, $\sum_{k=1}^{n} |I'_k \cap J_k| = \sum_{k=1}^{n} |I_k \cap J_k| + 1$, because $\{e\}$ now found the common element. So if we initially start with maximum $\sum |I_k \cap J_k|$ we are done.

Matroid Union. Proof of Theorem 2

Assume $X = E$ (Otherwise pick $M_1 \setminus (E - X), M_2 \setminus (E - X) \cdots$).

Copy the matroid in $E_1, E_2, \cdots, E_n$. Let $N_1 =$ matroid of $E_1 \cup E_2 \cup \cdots$ Such that X is indep in N_1 iff $X \cap E_i$ is indep in M_i for all i.

Let $N_2 =$ partition matroid that X is independent iff no two copies of same element are in X. Then, the maximum intersection is the independent set of matroid union.

Matroid Union. Proof of Theorem 2

By matroid intersection theorem, this value is:

$min_{Y_i \subseteq E_i}((r_1(Y_1) + r_2(Y_2) + \ldots + r_n(Y_n)) + | \cup_{i=1}^{n} (E_i - Y_i)|)$

By replacing Y_i into $Y_1 \cap Y_2 \ldots \cap Y_n$, we get $min_{Y \subseteq E}(r_1(Y) + r_2(Y) + \ldots + |E - Y|)$.

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By replacing Y_i into $Y_1 \cap Y_2 \ldots \cap Y_n$, we get $min_{Y \subseteq E}(r_1(Y) + r_2(Y) + \ldots + |E - Y|)$.

This models Matroid Union as a instance of matroid intersection.

Even though Matroid Union is just a matroid, it's not easy to implement an efficient oracle. So, this gives a simpliest poly-time algorithm.

Why

Why are any of these useful?

Disjoint spanning tree problem: Given a graph and an integer k, find k disjoint forests with maximum sum of sizes.

This is related to:

1. Shannon's switching game, Generic rigidity on plane.
2. Disjoint spanning trees can be efficiently solved, and they are related to graph cuts.

Arboricity is a related parameter: It is a minimum k such that the edges can be covered with k disjoint forests.

Algorithms for Matroid Union

Given n matroids in same ground set, find a maximal independent set in $M_1 \cup M_2 \cup \cdots M_n$.

It is enough to find, given n sets $X_1, X_2, \cdots, X_n$ where $X_i \subseteq I_i, X_i \cap X_j = \emptyset$, find $s \notin \cup X_i$ such that $(\cup X_i) \cup \{s\}$ is independent in $M_1 \cup M_2 \cup \cdots, M_n$ if it exists.

Let $D_{M_i}(X_i)$ be a directed graph on E where $\{(x, y) | x \in X_i, y \notin X_i, X_i - \{x\} + \{y\} \in I_i\}$. (Recall matroid intersection). Let $F_i = \{x \notin X_i | X_i \cup \{x\} \in I_i\}$. Let $X = \cup X_i, F = \cup F_i$.

Lemma. Let $s \in E - X$, $X \cup \{s\}$ is independent in $M_1 \cup \cdots M_n \iff D$ has a directed path from $(\cup F_i)$ to s.

Proof. ($\rightarrow$) Suppose D has no path from F to s, let T be the set of vertices having a directed path to S. $F \cap T = \emptyset$, and no edge from $E - T$ leads to T.

We claim that $X_i \cap T$ spans T for each i. Assume $t \in T, t \notin X_i$. Then $X_i \cup \{t\}$ is dependent in M_i (because $t \notin F_i$). Thus it includes a unique circuit C s.t. $t \in C$.

C has no element in $E - T$ because otherwise D_{M_i} has an edge from $E - T$ to T. $C \subseteq (X_i \cap T) \cup \{t\}$. $X_i \cap T$ spans t. $r_i(X_i \cap T) = r_i(T)$, $\sum r_i(T) = \sum r_i(X_i \cap T) \leq \sum |X_i \cap T| = |X \cap T|$. If $(X \cap T) \cup \{s\}$ is independent, then $|X \cap T| + 1 = \sum |Y_i| = \sum r_i(Y_i) \leq \sum r_i(T)$.

($\leftarrow$) Let $P = \{v_0, v_1, \cdots, v_p\}$ be a shortest path from $F = \cup_{i=1}^n F_i$ to s. We may assume $v_0 \in F_i$. Let $N_i = \{v_j | v_j v_{j+1} \in D_{M_i}(X_i)\}, N_i' = \{v_{j+1} | v_j v_{j+1} \in D_{M_i}(X_i)\}$. Let $Y_i = (X_i - N_i) \cup N_i'$ if $i > 1$, $Y_i = (X_i + \{v_0\}) - N_i - N_i'$ if $i = 1$. Then we claim Y_i is independent in M_i'.

Let $N_i = \{v_{i_1}, \cdots, v_{i_k}\}$, $N_i' = \{v_{i_1+1}, \cdots, v_{i_k+1}\}$. For each v_{i_l+1}, $(X_i - \{v_{i_k}\}) \cup \{v_{i_l+1}\}$ is independent in M_i. $X_i \cap \{v_{i_l+1}\}$ is dependent in M_i (because P is shortest). Thus, there is a unique circuit $C_l \subseteq X_i \cup \{v_{i_l+1}\}$ in M_i. C does not contain $v_{i_1}, \cdots, v_{i_{(l-1)}}$ (because P is shortest).

We know that X_i is independent, we can also see $(X_i - \{v_{i_1}\}) \cup \{v_{i_1+1}\}$ is independent in M_i. $(X_i - \{v_{i_1}, v_{i_2}\}) \cup \{v_{i_1+1}, v_{i_2+1}\}$ and so on.