Problem 1

You have to pick N vectors among all vectors in a way such that:

- For each "bag", there should be at most 1 vectors selected (Partition)
- Chosen vectors are linearly independent. (Linear)

Both conditions form a matroid. We can find the maximum size set which satisfies both condition with matroid intersection.

If the maximum set have size N, then we found the way. Otherwise we can see it is impossible.

Problem 2

You have to pick $N - 1$ edges among all edges in a way such that:

- For each "color", there should be at most 1 edge selected (Partition)
- Edges don't form a cycle (Graphic)

Both conditions form a matroid. We can find the maximum size set which satisfies both condition with matroid intersection.

If the maximum set have size $N - 1$, then we found the way. Otherwise we can see it is impossible.

Problem 3

You can create a graph for each of the friends, and say each edge encodes two different edges for the respective graph. Green edges affect both graph equally, red/blue edge remain equal for the visible ones, and is ineffective (say, it is a loop) for the not-visible ones.

You should find a minimum weight subset of edges such that:

- For G_1, the selected edges form a connected graph.
- For G_2, the selected edges form a connected graph.

Connectedness is not a matroid property, but it's dual is a matroid property. So:

- For G_1, the complement of selected edges are acyclic.
- For G_2, the complement of selected edges are acyclic.

Therefore, you can think the problem as "removing a maximum subset of weight" and taking a complement. This is an intersection of two dual graphic matroid.

Problem 4

Theorem. We can find a desired spanning tree if and only if there exists an acyclic edge subset where every vertex in L has a degree exactly 2.

Proof. $\leftarrow$: Since it is an independent set of graphic matroid, we can expand it to a base, which is spanning tree. Vertices with degree at least 2 are not leaves.

$\rightarrow$: L is independent, and it has degree at least 2 in the desired spanning tree. Thus, we can pick any two incident edges for each vertex in L.

You have to find any subset of edges among all edges in a way such that:

- For each vertex in L, there should be exactly 2 edge selected (Partition)
- Edges don't form a cycle (Graphic)

While we have the "exactly" condition (which is not a matroid), we don't have a limit on the number of edges, so we can relax the above condition as:

- For each vertex in L, there should be at most 2 edge selected (Partition)
- Edges don't form a cycle (Graphic)

Both conditions form a matroid. We can find the **maximum** size set which satisfies both condition with matroid intersection.

If the maximum set have size $2|L|$, then we found the way. Otherwise we can see it is impossible.

Problem 5

Take two partition matroid. Then for the intersection,
$max_{I \in I_1 \cap I_2} |I| = min_{X \subseteq E}(r_1(X) + r_2(E - X))$.

First term is the maximum matching by definition.

For the second term, $r_1(X) + r_2(E - X)$ is at least the MVC. We can take the vertices which increased the ranks: X will be covered by the left vertices that increased the rank, and $E - X$ will be covered by right vertices. Also, for a minimum vertex cover, we can classify for each edge whether it's left or right part is covered (if both are covered, take anything) and find a partition, which then will give the value $r_1(X) + r_2(E - X)$ same as MVC.

Problem 6

Recall Matroid Union Theorem:

Theorem 1. Let $M_1, M_2, \cdots, M_n$ be a matroids in E. Let I_i be a set of independent sets of M_i. Let $I = \{J_1 \cup J_2 \cup \cdots J_n : J_i \in I_i\}$. Then $M = (E, I)$ is a matroid, and the rank function of M is $r_M(X) = min_{Y \subseteq X}(r_1(Y) + r_2(Y) + \cdots + r_n(Y) + |X - Y|)$

6A

Let $M_1, M_2, \ldots, M_k$ a copy of matroids. M has k disjoint bases if and only if their union matroid has rank at least $kr(E)$. By Theorem 1, union matroid has rank $kr(E)$ if and only if $min_{X \subseteq E} kr(X) + |E - X| = kr(E)$, by definition. Now the proof is trivial by definition of min.

6B

$\rightarrow$: Each spanning tree contributes to at least $s-1$ edges in $C(P)$: Otherwise we can find isolated component.

$\leftarrow$: We will prove that $kr(X) + |E - X| \geq kr(E)$. Note that we only have to prove the result for X such that, there exists no e which $r(X \cup e) > r(X)$ (such X are called as *flat*). Now we can define a partition from X where two vertices are in same partition (equivalence class) iff there exists a path between two vertices that is a subset of X. There exists $N - r(X)$ partitions. Now we can see $|E - X| \geq k(r(E) + 1 - r(X) - 1)$.

6C

In a 2k-connected graph, $C(P)$ is at least $\frac{2ks}{2}$. This is because, the cut between $V(G) - P_i$ and P_i is at least $2k$. So we can compute the sum of them, and divide by two because every edge in $C(P)$ contributed to that quantity exactly twice (and not else). This is clearly greater than $(s-1)k$.