매트로이드와 그리디 알고리즘

Matroids and Greedy Algorithms

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KAIST School of Freshman
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Today's Topic

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- Greedy algorithms

Preliminaries

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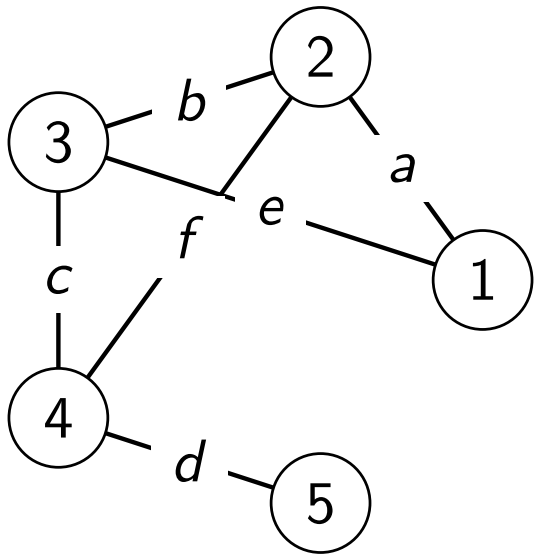

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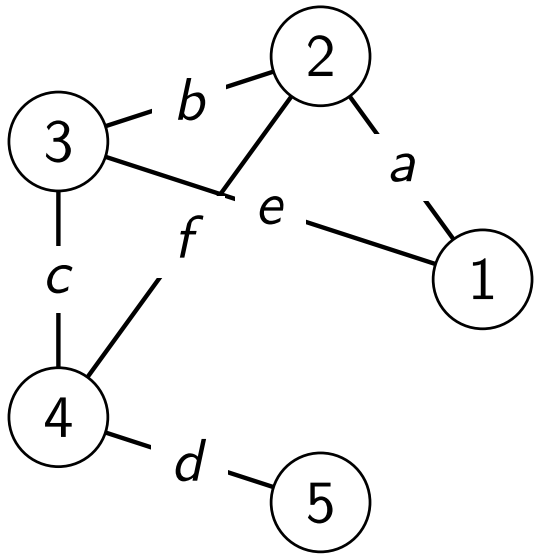

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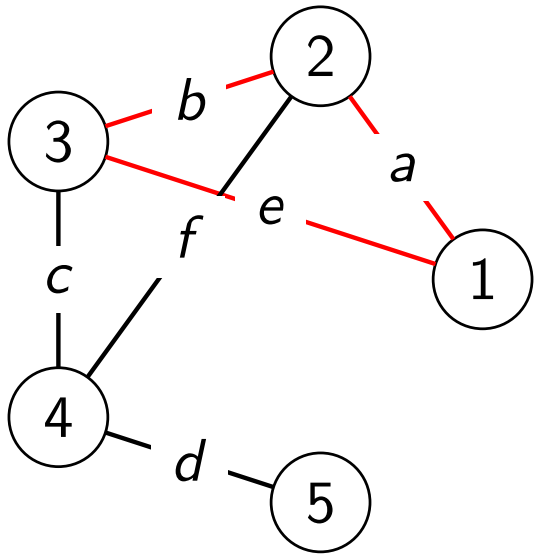

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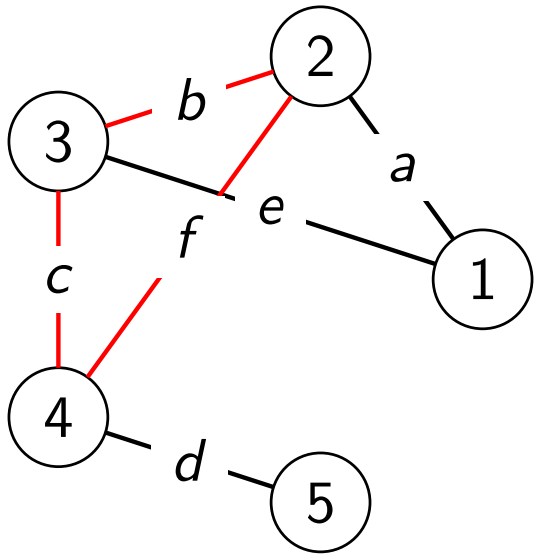

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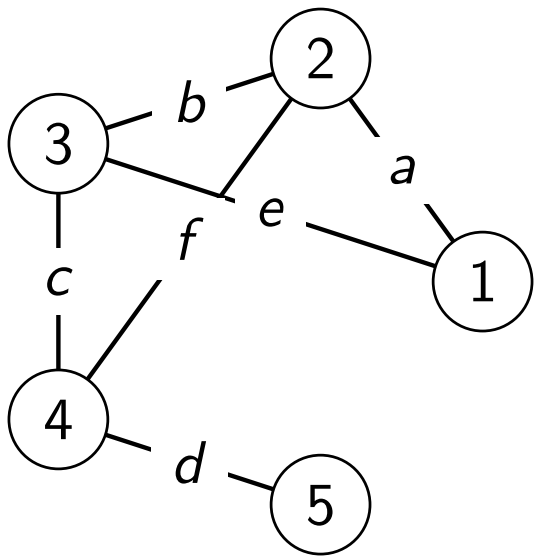

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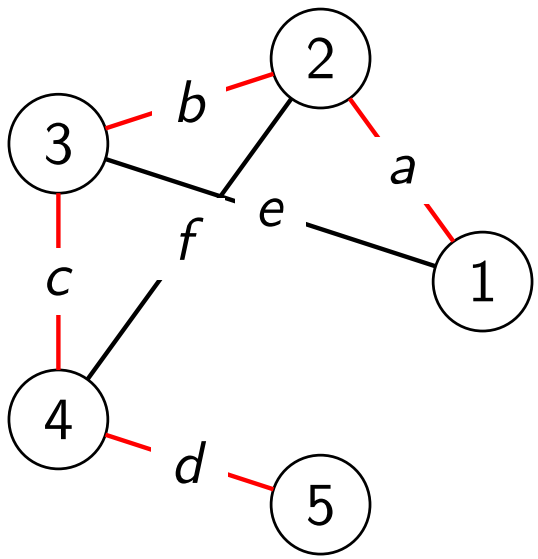

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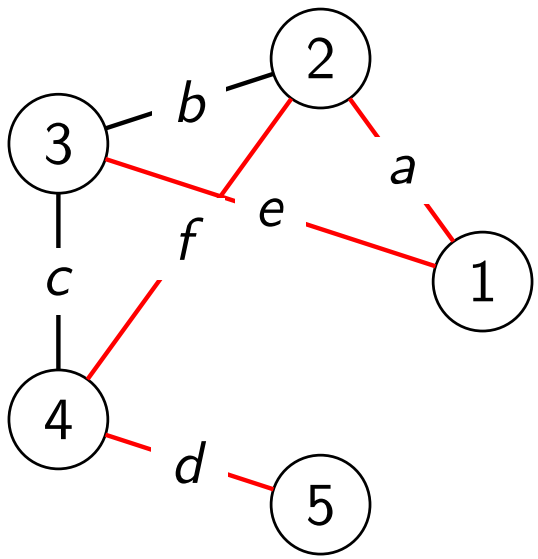

Section 1

What is a matroid?

Example from matrix

Consider:

- $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 3 \\ 2 & 4 & 2 & 6 \end{bmatrix}$ over the field $\mathbb{R}$.
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- $\emptyset$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\cdots$: linearly independent
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There is a strange fact:

- All the bases(=maximal independent sets) have the same size.

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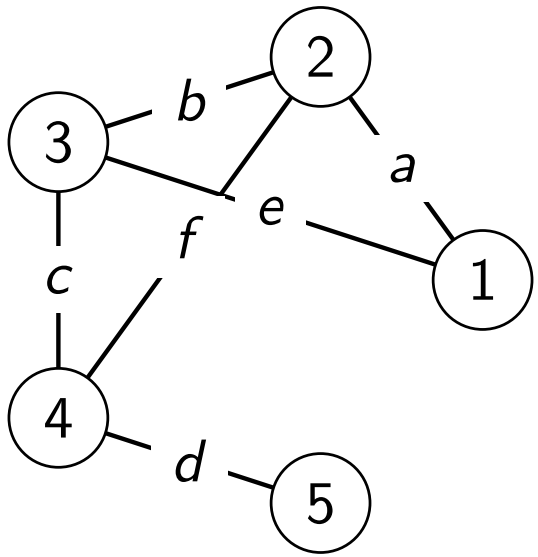

- $\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, c, d, f\}$: acyclic
- $\{a, b, c, e\}$, $\{a, b, e\}$, $\{a, b, c, f\}$: contains cycle

Example from graph

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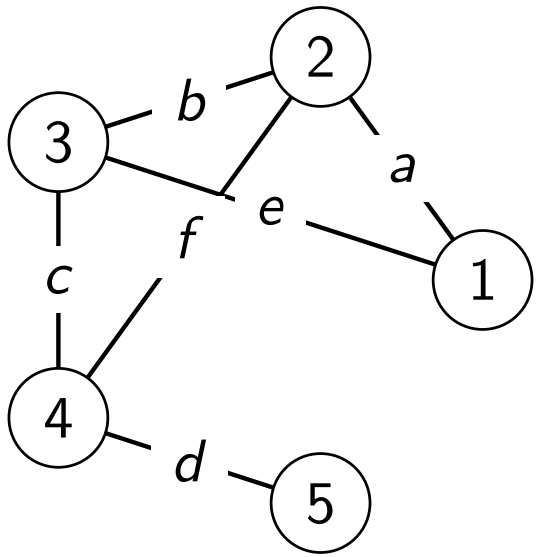

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- All the maximal acyclic subset(i.e. spanning tree) of E have the same size.

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Can we find a common generalization of those examples?

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Matroid [Whitney, 1935]

A matroid is a pair $M = (S, \mathcal{I})$ of a finite set S and a set $\mathcal{I} \subseteq 2^S$ satisfying:

(I1) $\emptyset \in \mathcal{I}$

(I2) If $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$

(I3) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then $X \cup \{e\} \in \mathcal{I}$ for some $e \in Y - X$.

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(I3) looks somewhat similar to following theorem from linear algebra.

(a) If S is a linearly independent set, and if $\mathbf{v} \in V$ but $\mathbf{v} \notin \text{span}(S)$, then the set $S \cup \{\mathbf{v}\}$ is linearly independent.

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We can easily obtain following property from (I3).

Property

All the maximal *independent* set have the same size, and they are called the *bases*. We denote the set of bases as $\mathcal{B}$.

Example of matroids

Uniform matroids

For $n \geq r \geq 0$, $U_{r,n}$ is a matroid on $[n] = \{1, 2, \cdots, n\}$ such that X is independent if and only if $|X| \leq r$.

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Vector matroid

For a matrix A over the field $\mathbb{F}$, the *vector matroid* $M[A]$ is the matroid on C, the column set of A, where $X \subseteq C$ is independent if and only if $A[X]$ is linearly independent.

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Note that:

- If $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$, then $A[\{1, 2, 4\}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$

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We can easily check that the base of $M[A]$ corresponds to the column base of A.

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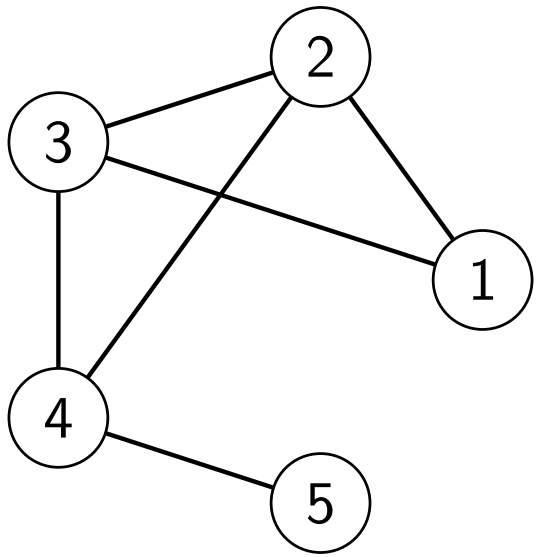
Cycle matroid

For a graph $G = (V, E)$, the *Cycle matroid* $M(G)$ is a matroid on E s.t. $X \subseteq E$ is independent if X contains no cycle.

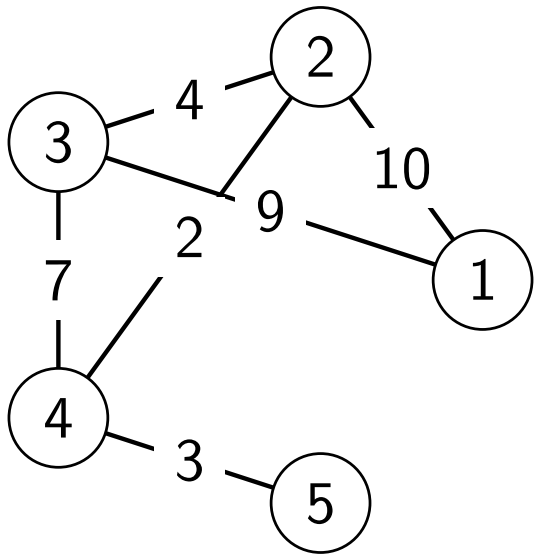
Graph vs. Weighted Graph

Graph $G = (V, E)$



Graph vs. Weighted Graph

Weighted graph $G = (V, E, w)$, $w : E \to \mathbb{R}$



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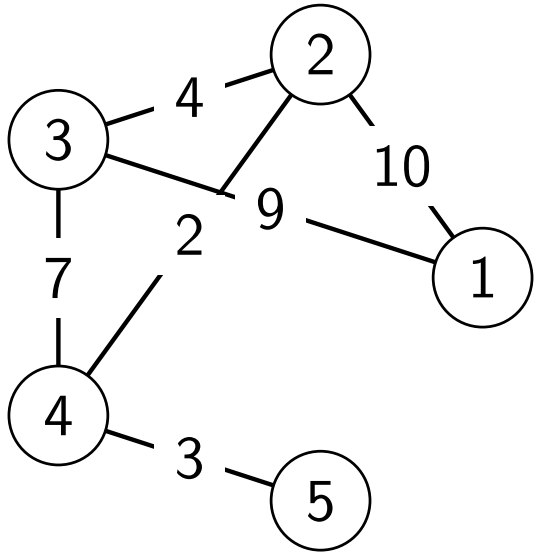
Example of weighted matroids

Consider following graph:



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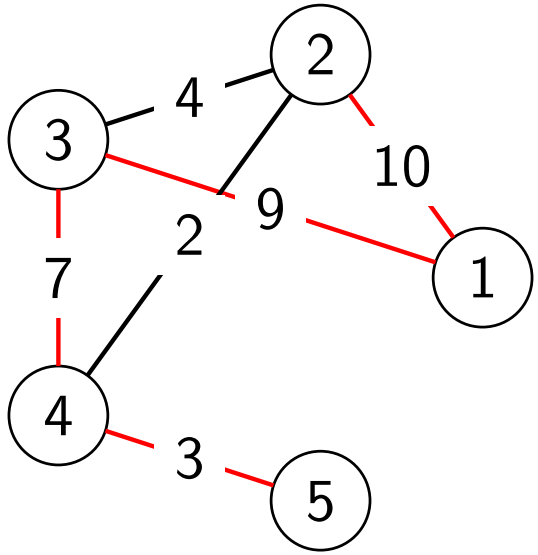

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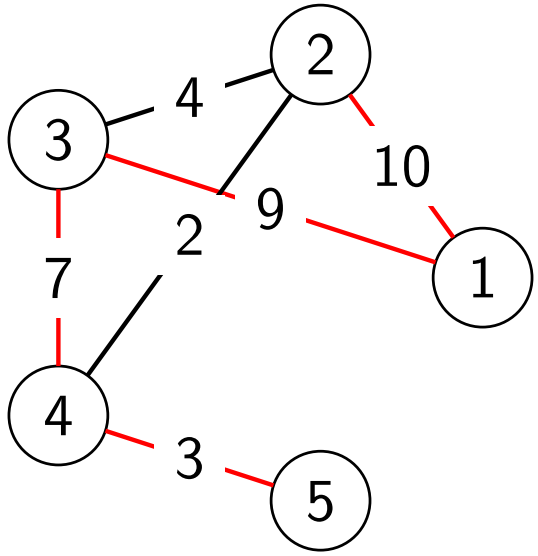


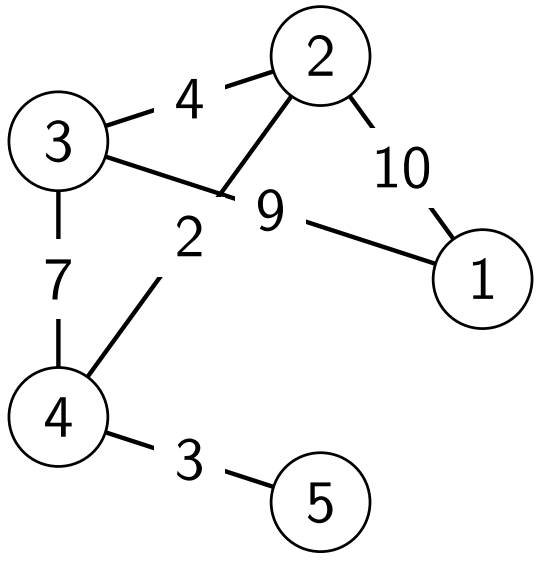
- $\mathcal{I}^\star$ is the set of maximum-weight spanning trees
- $w(\mathcal{I}^\star)$ is the weight of maximum spanning tree

Section 2

Greedy algorithms

Spanning trees and greedy algorithms

Recall the previous example.



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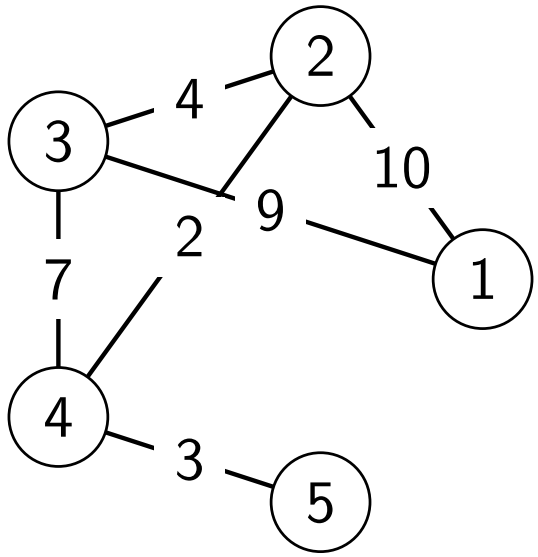

- How can we find maximum/minimum weighted spanning trees?

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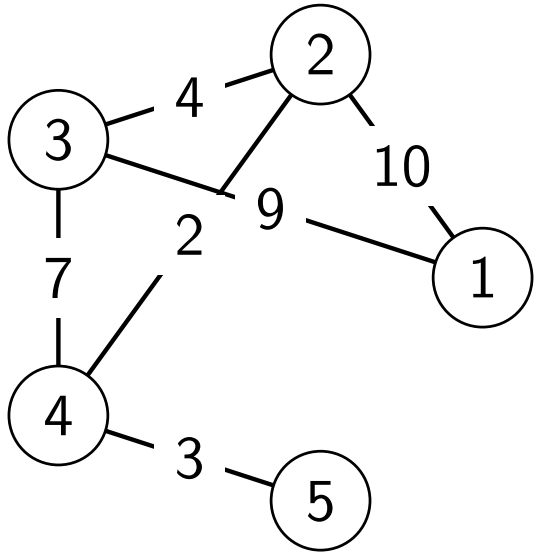

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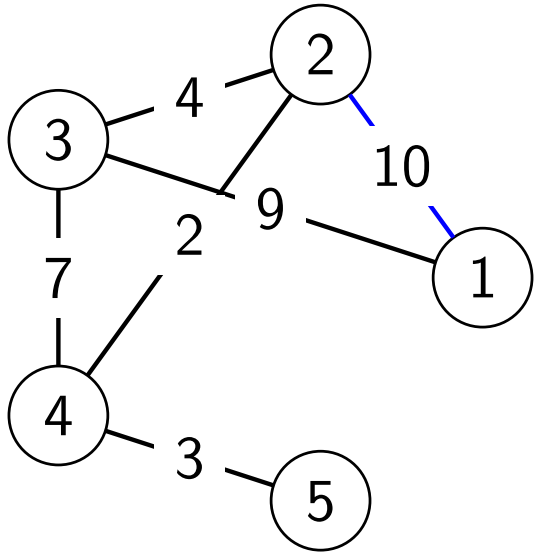

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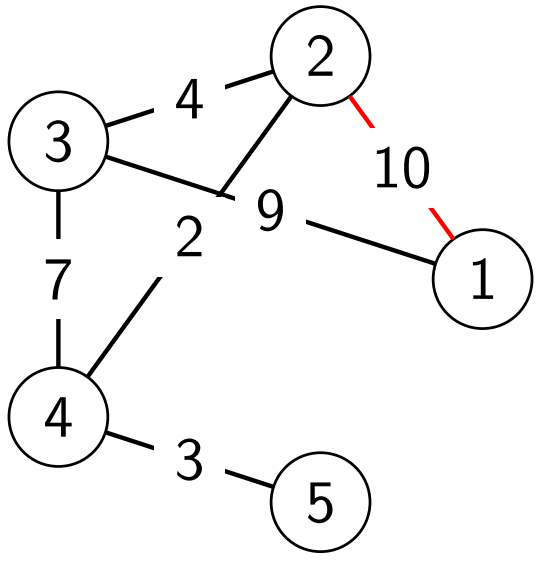

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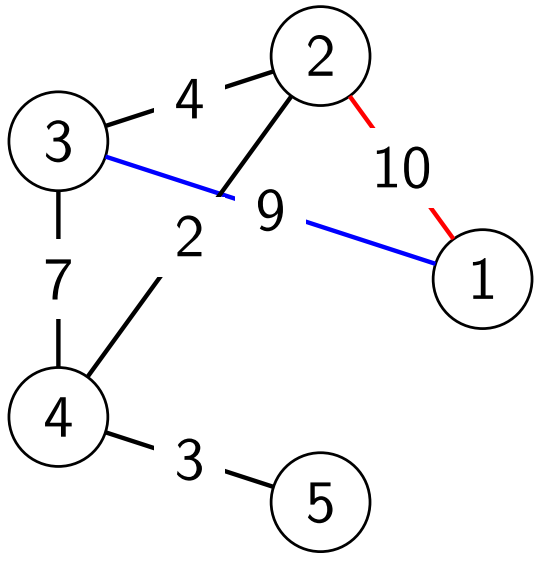

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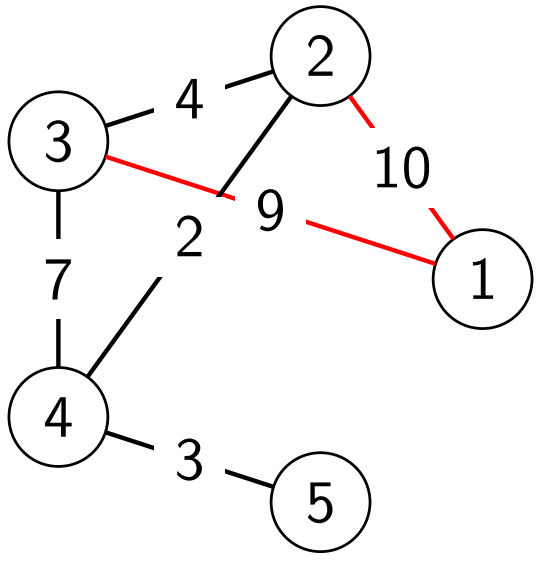

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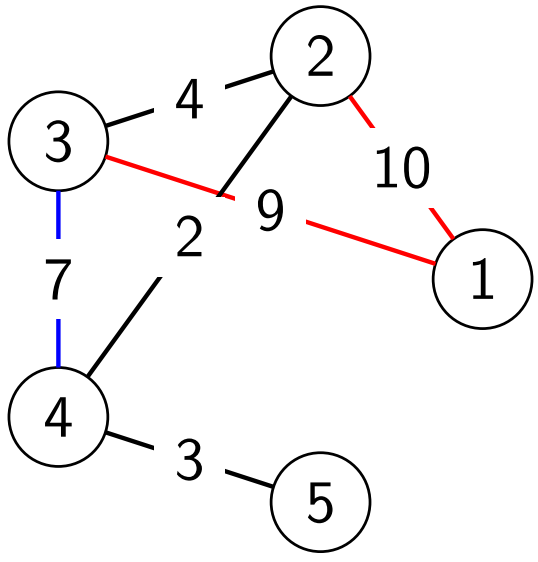

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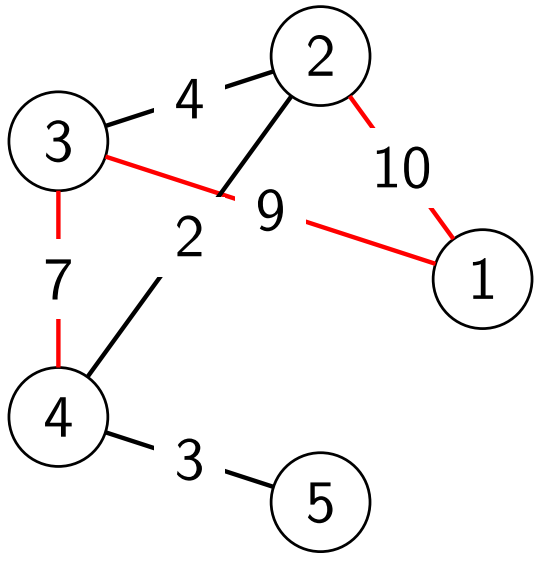

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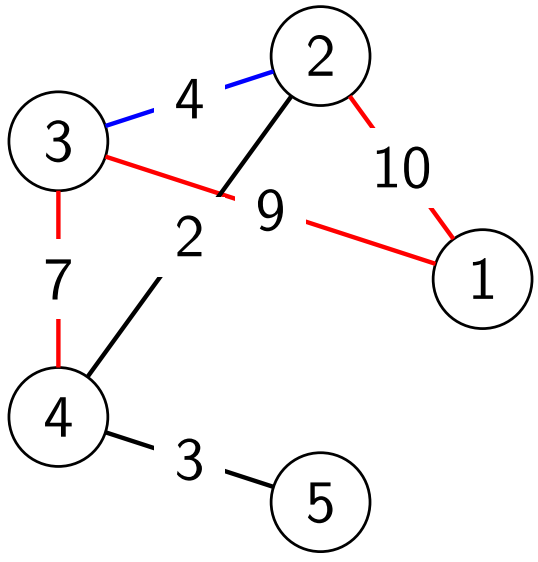

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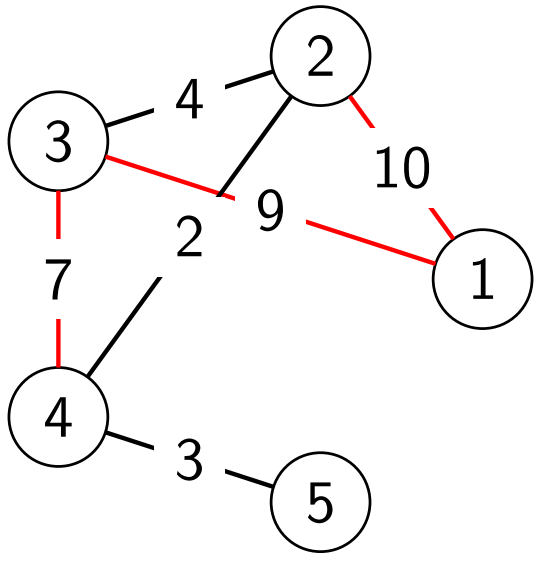

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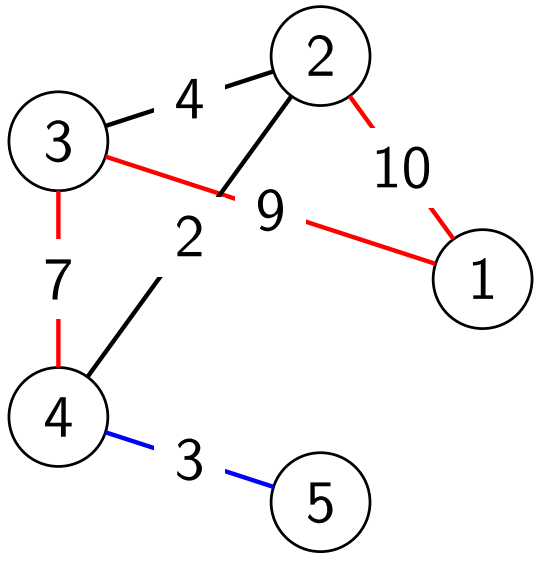

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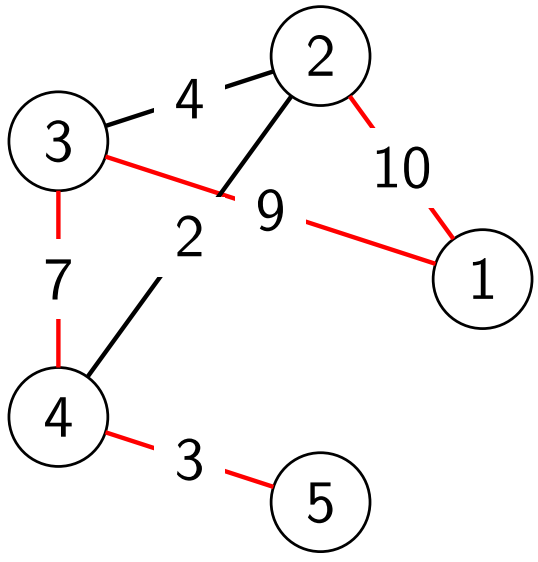

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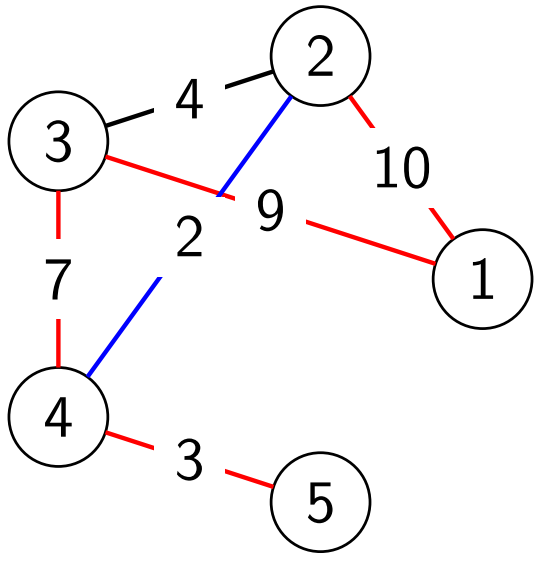

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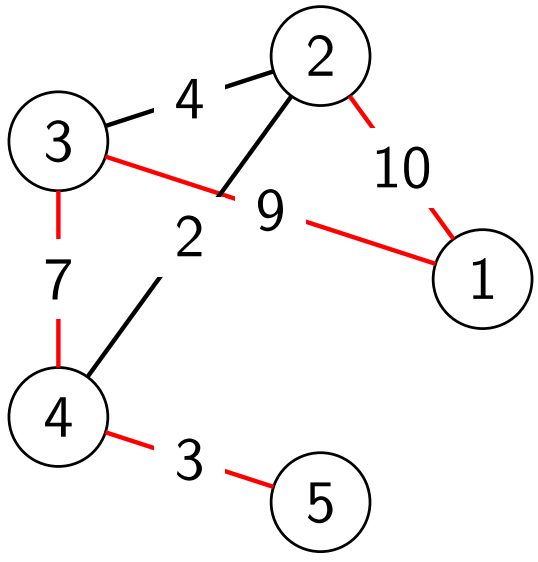

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  - if $T \cup \{e_i\}$ contains no cycle
    - $T := T \cup \{e_i\}$
- return T

Above algorithm finds an element of $\mathcal{I}^\star$ of a cycle matroid. Can we generalize it more general?

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So, is this algorithm *actually* works (correctly) for any matroid?

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- Let's prove it!

Correctness Proof

For sake of convenience, sort $S = \{x_1, \cdots, x_n\}$ so that $w(x_1) \geq \cdots \geq w(x_n)$.

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Algorithm to construct A

- $A := \{x_i\}$ initially
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- Then, $A = B \cup \{x_i\} - \{x_j\}$ for some $j > i$ so that

$$w(I^\star) \geq w(A) = w(B) - w(x_j) + w(x_i) \geq w(B) = w(I^\star).$$

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For $x \in S$, if $\{x\} \notin \mathcal{I}$, then no element of $\mathcal{I}$ contains x.

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Its proof is clear from (I2).

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- $S_x = \{y \in S - \{x\} : \{x, y\} \in \mathcal{I}\}$
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Lemma 3 - Optimal substructure property

Let $M = (S, \mathcal{I}, w)$ be a weighted matroid and suppose $x \in S$ satisfies:

- x is contained by some element of $\mathcal{I}^\star$.

Then, for each $B \in \mathcal{I}_x^\star$, $B \cup \{x\} \in \mathcal{I}^\star$.

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Combining those things all together, we can show the algorithm works correctly.

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We have shown that below algorithm works correctly for every (weighted) matroid.

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- for ($i := 1$ to n) do
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- Furthermore, it is efficient since it only takes $\mathcal{O}(n \log n + nf(n))$ time where $f(n)$ is running time of $(\star)$.

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- Q. Can we use matroid to formulate problems that *generic greedy algorithm* works correctly?

Defining matroid from greedy algorithms

Theorem

Suppose that we are given a finite set S and $\mathcal{I} \subseteq 2^S$ satisfying (I1) $\emptyset \in \mathcal{I}$ and (I2) Subset of independent set is independent. If (G) holds, $M = (S, \mathcal{I})$ is a matroid.

(G) Generic greedy algorithm works correctly for every $w : S \to \mathbb{R}$.

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Suppose that we are given a finite set S and $\mathcal{I} \subseteq 2^S$ satisfying (I1) and (I2). If (G) holds, $M = (S, \mathcal{I})$ is a matroid.

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  - That is, $X \cup \{e\} \notin \mathcal{I}$ for every $e \in Y - X$.

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- Suppose that (I3) does not holds for some $X, Y \in \mathcal{I}$.
- For sufficient $0 < \epsilon < (|Y| - |X|)/|X - Y|$, define

$$w(x) = \begin{cases} 1 + \epsilon & \text{if } x \in X \\ 1 & \text{if } x \in Y - X \\ 0 & \text{otherwise} \end{cases}.$$

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- Then, X', result of greedy algorithm contains X so that $|Y| + \epsilon|X \cap Y| = w(Y) \leq w(X') = (1+\epsilon)|X|$, which implies $\epsilon|X - Y| \geq |Y| - |X|$.

Results

Interestingly, although matroid is the concept that defines (linear) *independence* combinatorially, we checked that it also characterizes generic greedy algorithm.

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References

- James Oxley, *Matroid theory*, second ed., Oxford, 2011
- James Oxley, *What is a matroid?*, `https://www.math.lsu.edu/~oxley/survey4.pdf`