⟨Chapter 2. DETERMINANTS (SECTION 1. THE DETERMINANT OF A MATRIX⟩

∘ 1×1 행렬의 행렬식

• $A = (a)$, $(a \neq 0)$ 일 경우, $det(A) = a$ 이고, nonsingular 상태이다.

∘ 2×2 행렬의 행렬식

• $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ 에서 row operation 을 했을 때, pivot $\neq 0$ 이면 nonsingular 상태이다.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11}a_{21} & a_{12}a_{21} \\ a_{11}a_{21} & a_{11}a_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

∴ $a_{11}a_{22} - a_{12}a_{21} \neq 0$ 일때 nonsingular 상태이다.

따라서 $det(A) = a_{11}a_{22} - a_{12}a_{21}$

∘ $n \times n$ $(n \geq 3)$ 행렬의 행렬식

• 차수가 높아지면서 minor 와 cofactor 의 개념이 도입됨

• minor - 기호는 M_{ij}

ex) $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{array}{l} \\ \\ \to i^{th}\ row \\ \\ \end{array}$ $\to j^{th}$ column 에서 M_{ij}는 A에서 i^{th} row 와 j^{th} column 을 제거한 행렬이 된다.

ex) 3×3 행렬에서 $M_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$ 이다.

• cofactor는 다음과 같이 정의 된다.

$$A_{ij} = (-1)^{i+j} det(M_{ij})$$

• $n \times n$ 행렬의 행렬식은 다음과 같이 정의 된다.

$$det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

⇒ 3×3 행렬의 행렬식 구하기

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= (-1)^2 a_{11} det(M_{11}) + (-1)^3 a_{12} det(M_{12}) + (-1)^4 a_{13} det(M_{13})$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

- 2x2 행렬의 행렬식을 cofactor를 이용하여 구해보기

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\det(A) = a_{11}A_{11} + a_{12}A_{12} = (-1)^2 a_{11}\det(M_{11}) + (-1)^3 a_{12}\det(M_{12})$

$= a_{11}a_{22} - a_{12}a_{21}$

⇒ 위에서 operation 했던 결과의 값과 동일함.

- 행렬식의 기호 ⇒ $\det(A) = |A|$

- EXERCISES / p.96 1. Given $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{pmatrix}$

(a) Find the values of $\det(M_{21})$, $\det(M_{22})$, and $\det(M_{23})$.

- $M_{21} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$ $M_{22} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$ $M_{23} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

- $\det(M_{21}) = 2\cdot 2 - 4\cdot 3 = -8$
- $\det(M_{22}) = 3\cdot 2 - 4\cdot 2 = -2$
- $\det(M_{23}) = 3\cdot 3 - 2\cdot 2 = 5$

(b) Find the values of A_{21}, A_{22}, and A_{23}.

- $A_{21} = (-1)^{2+1}\det(M_{21}) = 8$
- $A_{22} = (-1)^{2+2}\det(M_{22}) = -2$
- $A_{23} = (-1)^{2+3}\det(M_{23}) = -5$

(c) Use your answers from part of (b) to compute $\det(A)$.

- $\det(A) = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$= (-1)\cdot(-8) + (-2)\cdot(-2) + 3\cdot(-5)$

$= -3$

- EXERCISES / p.97 4-(c). Evaluate the following determinants by inspection

$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= a_{11}(-1)^{1+1}\det(M_{11}) + a_{12}(-1)^{1+2}\det(M_{12}) + a_{13}(-1)^{1+3}\det(M_{13})$

$= 3\cdot 1\cdot(1.2 - 1.2) + 0 + 0$

$= 0$

- EXERCISES / p.98 11. Let A and B be 2×2 matrices.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

(a) Does $\det(A+B) = \det(A) + \det(B)$?

- $A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$
- $\det(A+B) = (a_{11}+b_{11})(a_{22}+b_{22}) - (a_{12}+b_{12})(a_{21}+b_{21}) =$
- $\det(A) = a_{11}a_{22} - a_{12}a_{21}$
- $\det(B) = b_{11}b_{22} - b_{12}b_{21}$

$\Rightarrow \det(A+B) \neq \det(A) + \det(B)$

(b) Does $\det(AB) = \det(A)\det(B)$?

- $AB = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$
- $\det(AB) = (a_{11}b_{11}+a_{12}b_{21})(a_{21}b_{12}+a_{22}b_{22}) - (a_{11}b_{12}+a_{12}b_{22})(a_{21}b_{11}+a_{22}b_{21})$
  $= a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} - a_{21}b_{11}a_{12}b_{22} - a_{22}b_{21}a_{11}b_{12}$
- $\det(A)\det(B) = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$
  $= a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} - a_{21}a_{12}b_{11}b_{22} - a_{11}a_{22}b_{21}b_{12}$

$\Rightarrow \det(AB) = \det(A)\det(B)$

(c) Does $\det(AB) = \det(BA)$?

- (b) 번에서 $\det(AB) = \det(A)\det(B)$ 임을 찾았으므로, determinant는 scalar 값이기 때문에 $\det(A)\det(B) = \det(B)\det(A)$ 이고, 이는 $\det(BA)$ 이므로, $\det(AB) = \det(BA)$ 이다.

< Chapter 2. DETERMINANTS
SECTION 2. PROPERTIES OF DETERMINANTS >

◦ $n\times n$ 행렬의 행렬식의 표현법

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$
$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

⇒ 어떤 행이나 어떤 열을 기준으로 하더라도 determinant의 값은 항상 같다.

∴ '0'이 많이 포함되어 있는 열이나 행을 기준으로 하면 계산이 편해짐

⇓

ex) $\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 3 \end{vmatrix} = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + a_{41}A_{41}$

$$= 0\cdot A_{11} + 0\cdot A_{21} + 0\cdot A_{31} + 2\cdot A_{41}$$
$$= 2\cdot(-1)^{4+1}\det(M_{41})$$
$$= -2\cdot\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix}$$
$$= -2\cdot 3\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 12$$

◦ $n\times n$ 행렬에서 $\det(A) = \det(A^T)$ 이다.

• 위의 성질에서 기준은 행, 렬. 모두 상관 없으므로 성립한다.

◦ $n\times n$ 행렬에서 어떤 행이나 열의 성분 모두 '0'이면 $\det(A)=0$ 이다.

• $\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ 에서 $a_{i1} = a_{i2} = \cdots = a_{in} = 0$ 이기 때문.

◦ $n\times n$ 행렬에서 두 행이나 두 열이 같다면 $\det(A) = 0$ 이다.

• 행렬식의 연산을 recursive 하게 최종 같은 두행이나 열만 남기면, 그 determinant가 0 이기 때문이다.

◦ $a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

• A 행렬의 j^{th} row의 성분을 i^{th} row 성분으로 바꾼 행렬을 A^* 이라하자.

이때 $\det(A^*) = a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn}$

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}$$
$$= 0$$

∵ A^*은 row 2개가 일치하기 때문. A^*에서 2번째 두개의 row가 같을 때, 같은 row 중 하나를 기준으로 행렬식을 구하면 $\det(A) = \det(A^*)$ 이므로 $\det(A) = 0$ 이다.

∘ $n \times n$ 행렬 A에서 2개의 row가 서로 위치가 바뀐 행렬을 EA라 할 때,

$\det(EA) = -\det(A)$ 이다.

E: Elementary Matrix type I.

• 2x2 행렬에서의 관찰

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $EA = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$

$\Rightarrow \det(A) = a_{11}a_{22} - a_{12}a_{21}$

$\Rightarrow \det(EA) = a_{12}a_{21} - a_{11}a_{22} = -\det(A)$

• 3x3 행렬에서의 관찰

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $E_{13}A = \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}$

$\Rightarrow \det(A) = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

$\Rightarrow \det(E_{13}A) = -a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} + a_{22}\begin{vmatrix} a_{31} & a_{33} \\ a_{11} & a_{13} \end{vmatrix} - a_{23}\begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix}$

$= a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

$= -\det(A)$

• So, $\det(E_{ij}A) = -\det(A)$

• $\det(E_{ij}) = \det(E_{ij}I) = -\det(I) = -1$

∘ $n \times n$ 행렬에서 어느 한 row가 α배가 된 행렬을 EA라 할 때,

$\det(EA) = \alpha \det(A)$ 이다.

E: Elementary Matrix type II.

• proof) $\det(EA) = \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \cdots + \alpha a_{in}A_{in}$

$= \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in})$

$= \alpha \det(A)$

• $\det(E) = \det(EI) = \alpha \det(I) = \alpha$

• $\det(EA) = \alpha \det(A) = \det(E)\det(A)$

∘ $n \times n$ 행렬에서 어느 한 row에 다른 row의 상수배만큼 더해준 행렬을 EA라 할 때,

$\det(EA) = \det(A) = \det(E)\det(A)$ 이다.

E: Elementary Matrix type III.

• proof) $\det(EA) = (a_{i1}+ca_{j1})A_{i1} + (a_{i2}+ca_{j2})A_{i2} + \cdots + (a_{in}+ca_{jn})A_{in}$

$= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} + c\underbrace{(a_{j1}A_{i1} + a_{j2}A_{i2} + \cdots + a_{jn}A_{in})}_{\Downarrow}$

$= \det(A)$

앞 page 에서 '0' 임을 증명했음

• $\det(EA) = \det(A) = \det(E)\det(A)$

- $\therefore \det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$
- $\det(AE) = \det((AE)^T) = \det(E^TA^T) = \det(E^T)\det(A^T) = \det(E)\det(A)$
- $n \times n$ 행렬에서 'A가 singular 상태이다'와 '$\det(A) = 0$ 이다'는 필요충분 조건이 성립한다.
  - 'A가 singular 상태이면 $\det(A) = 0$ 이다' proof
    $\Rightarrow$ A가 singular 상태이면 역행렬을 구하는 과정 $E[A|I] \to [I|A^{-1}]$ 에서 EA 의 pivot이 0이 되는 경우가 발생한다. 이 경우 행렬의 한 row가 전부 0이 되어버리므로 $\det(EA) = \det(A) = 0$ 이다.
  - '$\det(A) = 0$ 이면 A는 singular 상태이다' proof
    $\Rightarrow$ $\det(A) = 0$ 인 경우는 A의 row중 같은 row가 있거나 row의 성분이 모두 0인 경우이다. 이때 $E[A|I] \to [I|A]$ 이 불가능 하므로 역행렬이 존재하지 않는다.
  - $\therefore$ 필요충분 조건이 성립한다.
- E^{-1} 의 determinants 구하기
  - type I 의 역행렬 = 원래 matrix와 같음
    ex) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\therefore \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 1 \cdot (-1) = -1$
  - type II 의 역행렬 = 원래 matrix가 α배 해주는 것이라면, 역행렬은 $\frac{1}{\alpha}$배 해주는 것임.
    ex) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\therefore \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \frac{1}{\alpha} = \frac{1}{\alpha}$
  - type III 의 역행렬 = 원래 matrix가 α배해서 더해주는 것이라면, 역행렬은 $-\alpha$배를 더해줌.
    ex) $\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\therefore \begin{vmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 = 1$
  - $\therefore \det(E^{-1}) = \dfrac{1}{\det(E)}$
- $n \times n$ 행렬에서 $\det(AB) = \det(A)\det(B)$ 이다.
  - B가 singular 일 때 proof) $\det(AB) = \det(0) = 0 = \det(A)\det(B)$
  - B가 nonsingular 일 때 proof) 어떤 행렬이 nonsingular 상태이면 그 행렬은 $E_k \cdot E_{k-1} \cdots E_1 \cdot I$ 처럼 elementary matrices와 identity matrix의 곱으로 나타낼 수 있다. ($\because$ 어떤 행렬의 역행렬을 구할 때 row operation이 전부 elementary matrix 이기 때문) 따라서 $\det(AB) = \det(A E_k \cdot E_{k-1} \cdots E_1)$ 이고 $\det(AE) = \det(A)\det(E)$ 이므로 $\det(AB) = \det(A) \cdot \det(E_k)\det(E_{k-1}) \cdots \det(E_1)$ $= \det(A)\det(E_k E_{k-1} \cdots E_1) = \det(A)\det(B)$
- $\therefore \det(AB) = \det(A)\det(B)$

∘ $n \times n$ 행렬에서 $\det(A^{-1}) = \dfrac{1}{\det(A)}$ 이다.

• A가 nonsingular 행렬이면 $A = E_k E_{k-1} \cdots E_1 I$ 로 나타낼 수 있고, $A^{-1} = I^{-1} E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ 이므로 $\det(A^{-1}) = \det(I^{-1}E_1^{-1}E_2^{-1}\cdots E_k^{-1}) = \det(I^{-1})\det(E_1^{-1})\det(E_2^{-1})\cdots\det(E_k^{-1}) =$

$$\frac{1}{\det(I)\det(E_1)\det(E_2)\cdots\det(E_k)} = \frac{1}{\det(I\cdot E_1\cdot E_2\cdots E_k)} = \frac{1}{\det(A)}$$ 이다.

• $\therefore \det(A^{-1}) = \dfrac{1}{\det(A)}$

∘ EXERCISES / p.103 1-(b). Evaluate each of the following determinants by inspection.

let) $A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix}$, $\quad \det(A) = \det(\underset{\substack{\downarrow \\ \text{Elementary matrix} \\ \text{type Ⅲ}}}{EA})$ 이므로 $\quad EA = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ 이고

EA는 triangular matrix 이므로 $\det(A) = 1\cdot3\cdot2\cdot5 = 30$

∘ EXERCISES / p.103 2-(a). Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & 3 \end{pmatrix}$, use elimination method to evaluate det(A)

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(3rd row = 2nd row ×2 + 3rd row
4th row = 4th row − 2th row

(4th row = 1st row − 4th row
change 1st row 2nd row

4th row = 4th row − 3rd row

결과 행렬이 triangular matrix 이므로 $\det(A) = 1\cdot1\cdot5\cdot2 = 10$

∵ Gaussian Elimination 사용되는 행렬은 모두 elementary matrix 이기 때문

∘ EXERCISES / p.104 4. Find all possible choices of c that would make the following matrix singular.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix},$$

$$\begin{aligned}
\det(A) &= 1\cdot A_{11} + 1\cdot A_{12} + 1\cdot A_{13} \\
&= (-1)^{1+1}\det(M_{11}) + (-1)^{1+2}\det(M_{12}) + (-1)^{1+3}\det(M_{13}) \\
&= \begin{vmatrix} 9 & c \\ c & 3 \end{vmatrix} - \begin{vmatrix} 1 & c \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 9 \\ 1 & c \end{vmatrix} \\
&= 27 - c^2 - 3 + c + c - 9 \\
&= -c^2 + 2c + 15 \\
&= -(c+3)(c-5)
\end{aligned}$$

$\therefore \det(A) = 0$ 일 때 singular 이므로, $c = -3$ or 5 이다.

∘ EXERCISES / p.104 7. Let A and B be 3×3 matrix with $\det(A) = 4$, $\det(B) = 5$, Find value of each Question.

(a) $\det(AB) = \det(A)\det(B) = 4\cdot5 = 20$

(b) $\det(3A) = \begin{vmatrix} 3&0&0\\0&1&0\\0&0&1 \end{vmatrix}\begin{vmatrix} 1&0&0\\0&3&0\\0&0&1 \end{vmatrix}\begin{vmatrix} 1&0&0\\0&1&0\\0&0&3 \end{vmatrix}\det(A) = 3^3 \times 4 = 108$

(c) $\det(2AB) = \det(2A)\det(B) = \begin{vmatrix} 2&0&0\\0&1&0\\0&0&1 \end{vmatrix}\begin{vmatrix} 1&0&0\\0&2&0\\0&0&1 \end{vmatrix}\begin{vmatrix} 1&0&0\\0&1&0\\0&0&2 \end{vmatrix}\det(A)\det(B) = 2^3 \times 4 \times 5 = 160$

(d) $\det(A^{-1}B) = \det(A^{-1})\det(B) = \dfrac{\det(B)}{\det(A)} = \dfrac{5}{4}$

< Chapter 2. DETERMINANTS (SECTION 3. CRAMER'S RULE >

∘ The Adjoint of a Matrix

- A^{-1}를 쉽게 구하기 위해 adjoint 행렬을 정의함

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$$\left(A_{ij} = (-1)^{i+j} \det(M_{ij})\right)$$

- $$A(\operatorname{adj} A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} = \begin{pmatrix} \det(A) & 0 & 0 \cdots & 0 \\ 0 & \det(A) & 0 \cdots & 0 \\ \vdots & & \det(A) & 0 \\ 0 & 0 & & \det(A) \end{pmatrix}$$

$$= \det(A) \cdot I$$

- $\therefore$ If $\det(A) \neq 0$ (A is nonsingular), $A\left(\frac{1}{\det(A)} \operatorname{adj} A\right) = I$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A$$

- 2×2 행렬의 역행렬

let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, A is nonsingular. $A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A$

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} = a_{11} \cdot (-1)^{1+1} \det(M_{11}) + a_{12}(-1)^{1+2} = a_{11}a_{22} - a_{12}a_{21}$$

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

∘ Crammer's Rule

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, $Ax = b$ 에서 A_i = obtained by replacing the i th column of A by b. If x is unique solution to $Ax=b$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \quad (i = 1, 2, \cdots, n)$$

proof) $x = A^{-1}b = \frac{1}{\det(A)}(\text{adj}\,A)\,b$

$$\Rightarrow x_i = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{\det(A)} = \frac{\det(A_i)}{\det(A)}$$

- EXERCISES / p.109 1-(c). Compute $\det(A)$, adj A, A^{-1}.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$

- $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}$$

$$= (-3) + (0) + 6$$

$$= 3$$

- adj A $= \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$

$$= \begin{pmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{pmatrix}$$

- A is nonsingular. So. $A^{-1} = \frac{1}{\det(A)}$ adj A

$$= \frac{1}{3}\begin{pmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & 5 \end{pmatrix}$$

- EXERCISES / p.110 2-(c). Use Cramer's rule to solve each of the following systems

$$\begin{cases} 2x_1 + x_2 - 3x_3 = 0 \\ 4x_1 + 5x_2 + x_3 = 8 \\ -2x_1 - x_2 + 4x_3 = 2 \end{cases} \Rightarrow \begin{pmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 2 \end{pmatrix}$$

let $A = \begin{pmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{pmatrix}$, $\det(A) = 2\begin{vmatrix} 5 & 1 \\ -1 & 4 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ -2 & 4 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ -2 & -1 \end{vmatrix} = 6$

So, A is nonsingular ($\because \det(A) \neq 0$)

let A_i is obtained by replacing the i column of A by b.

$A_1 = \begin{pmatrix} 0 & 1 & -3 \\ 8 & 5 & 1 \\ 2 & -1 & 4 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 & 0 & -3 \\ 4 & 8 & 1 \\ -2 & 2 & 4 \end{pmatrix}$, $A_3 = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & 8 \\ -2 & -1 & 2 \end{pmatrix}$

$\det(A_1) = 24$, $\det(A_2) = -12$, $\det(A_3) = 12$

$\therefore x_1 = \frac{24}{6} = 4$, $x_2 = \frac{-12}{6} = -2$, $x_3 = \frac{12}{6} = 2$

- EXERCISES / p. 110 6. If A is singular, what can you say about the product $A \operatorname{adj} A$?

$A \operatorname{adj} A = \det(A) \cdot I = 0$. $\det(A) = 0$ 이기 때문에, 양변을 $\det(A)$로 나누는 것은 불가능 함.

- EXERCISES / p. 110 8. Let A be a nonsingular $n \times n$ matrix with $n > 1$. Show that $\det(\operatorname{adj} A) = (\det(A))^{n-1}$

$A \operatorname{adj} A = \det(A) \cdot I$ 에서 양변에 det를 하면 $\det(A \operatorname{adj} A) = \det(\det(A) I)$

는 $\det(A) \det(\operatorname{adj} A) = \det(\det(A)) \det(I)$ 이다.

$\det(\det(A)) = (\det(A))^n$ 이고 $\det(I) = 1$ 이므로, $\det(\operatorname{adj} A) = (\det(A))^{n-1}$ 이다.

- EXERCISES / p. 110 9-(a). Let A be a 4×4 matrix. If $\operatorname{adj} A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 2 \end{pmatrix}$

- Calculate the value of $\det(\operatorname{adj} A)$. What should the value of $\det(A)$ be?

$\det(\operatorname{adj} A) = 2 \begin{vmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ -2 & -1 & 2 \end{vmatrix} = 8$, $\therefore \det(A) = 2$

⟨ Chapter 2. DETERMINANTS (CHAPTER TEST A, B ⟩

- A-1. $\det(AB) = \det(BA)$?

  $\Rightarrow$ True.

  A와 B가 singular인 상태이면 elementary matrix들의 곱으로 나타나고, 이는 서로 분리가 가능하기 때문.

- A-2. $\det(A+B) = \det(A) + \det(B)$?

  $\Rightarrow$ FALSE

  행렬의 덧셈 결과는 분리할 수 없음

- A-3. $\det(cA) = c\det(A)$?

  $\Rightarrow$ FALSE

  $\det(cA)$ = A가 $n\times n$ 행렬일 때, $\det(cA) = c^n\det(A)$ 이다.

- A-4. $\det((AB)^T) = \det(A)\det(B)$?

  $\Rightarrow$ True.

  $\det((AB)^T) = \det(AB) = \det(A)\det(B)$

- A-5. $\det(A) = \det(B)$ implies $A = B$?

  $\Rightarrow$ FALSE

  행렬식이 같은 경우는 무수히 많을 수 있다, 그러나 그 행렬들의 성분이 모두 같은 것은 아니다.

- A-6. $\det(A^k) = \det(A)^k$?

  $\Rightarrow$ True

  A-1 번과 이유가 같음

- B-1. Let A and B be 3x3 matrices with $\det(A) = 4$, $\det(B) = 6$ and let E be an elementary matrix of type I. Determine the value of each of the following.

  (a) $\det(\frac{1}{2}A) = \left(\frac{1}{2}\right)^3\det(A) = \frac{1}{2}$

  (b) $\det(B^{-1}A^T) = \det(B^{-1})\det(A^T) = \dfrac{\det(A)}{\det(B)} = \dfrac{2}{3}$

(c) $\det(EA^2) = \det(E)\det(A)\det(A) = \det(E)\det(A)^2 = -16$

◦ B-2. Given $A = \begin{pmatrix} x & 1 & 1 \\ 1 & x & -1 \\ -1 & -1 & x \end{pmatrix}$

(a) Compute the value of det(A) (Your answer should be a function of x)

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= x\begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ -1 & x \end{vmatrix} + \begin{vmatrix} 1 & x \\ -1 & -1 \end{vmatrix}$$

$$= x(x^2-1) - (x-1) + (x-1)$$

$$= x(x-1)(x+1)$$

(b) For what values of x will the matrix be singular? Explain.

(a) 번에서 $\det(A) = x(x-1)(x+1)$ 이므로 $x = 0, 1, -1$ 이면 $\det(A)=0$ 이고, $\det(A)=0$ 이면 singular 이다.

◦ B-3, Given $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$

(a) Compute the LU factorization of A.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Use the LU factorization to determine the value of det(A).

$$\det(A) = (1\cdot1\cdot1\cdot1)\cdot(1\cdot1\cdot1\cdot1) = 1$$

∘ B-4. If A is nonsingular $n\times n$ matrix, show that $A^T A$ is nonsingular and $\det(A^T A) > 0$.

$\det(A^T A) = \det(A^T)\det(A)$ 이고, $\det(A^T) = \det(A)$ 이므로 A^T is nonsingular,

$\det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A) = \det(A)^2$ 이다. $\therefore \det(A^T A) > 0$

∘ B-5. Let A be an $n\times n$ matrix. Show that if $B = S^{-1}AS$ for some nonsingular matrix S, then $\det(B) = \det(A)$.

$$\det(B) = \det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \frac{\det(S)}{\det(S)}\det(A) = \det(A)$$

∘ B-6. Let A and B be $n\times n$ matrices and let $C = AB$. Use determinants to show that if either A or B is singular, then C must be singular.

$\det(C) = \det(AB) = \det(A)\det(B)$ 이다. A or B가 singular 이면

$\det(A) = 0$ or $\det(B) = 0$ 이다. 따라서 $\det(C) = 0$ 이므로, C는 반드시 singular 이다.